The Verlinde Algebra is Twisted Equivariant K-Theory

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K-theory in various forms has recently received much attention in 10-dimensional superstring theory. Our raised consciousness about *twisted* K-theory led to the serendipitous discovery that it enters in a different way into 3-dimensional topological field theories, in particular Chern-Simons theory. Namely, as the title of the paper reports, the Verlinde algebra is a certain twisted K-theory group. This assertion, and its proof, is joint work with Michael Hopkins and Constantin Teleman. The general theorem and proof will be presented elsewhere [FHT]; our goal here is to explain some background, demonstrate the theorem in a simple nontrivial case, and motivate it through the connection with topological field theory.

From a mathematical point of view the Verlinde algebra is defined in the theory of loop groups. Let G be a compact Lie group. There is a version of the theorem for *any* compact group G, but here for the most part we focus on connected, simply connected, and simple groups—$G = SU_2$ is the simplest example. In this case a central extension of the free loop group LG is determined by the *level*, which is a positive integer k. There is a finite set of equivalence classes of positive energy representations of this central extension; let $V_k(G)$ denote the free abelian group they generate. One of the influences of 2-dimensional conformal field theory on the theory of loop groups is the construction of an algebra structure on $V_k(G)$, the *fusion product*. This is the *Verlinde algebra* [V].

Let G act on itself by conjugation. Then with our assumptions the equivariant cohomology group $H^3_G(G)$ is free of rank one. Let $h(G)$ be the dual Coxeter number of G, and define $\zeta(k) \in H^3_G(G)$ to be $k + h(G)$ times a generator. We will see that elements of H^3 may be used to *twist* K-theory, and so elements of equivariant H^3 twist equivariant K-theory.

Theorem (Freed-Hopkins-Teleman). *There is an isomorphism of algebras*

$$V_k(G) \cong K_G^{\dim G + \zeta(k)}(G),$$

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where the right hand side is the $\zeta(k)$-twisted equivariant K-theory in degree $\dim G$.

The group structure on the right-hand side is induced from the multiplication map $G \times G \to G$.

For an arbitrary compact Lie group G the level k is replaced by a class in $H^4(BG;\mathbb{Z})$ and the dual Coxeter number $h(G)$ is pulled back from a universal class in $H^4(BSO;\mathbb{Z})$ via the adjoint representation.[1] The twisting class is obtained from their sum by transgression.

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[1]The story is a bit more subtle and involves $H^2(BG;\mathbb{Z}/2\mathbb{Z})$—so a twisting using $H^1_G(G;\mathbb{Z}/2\mathbb{Z})$—as well, though we do not discuss that here.

§1 Twisted K-theory

Twistings of cohomology theories are most familiar for ordinary cohomology. Let M be a manifold (or suitably nice space). Then a flat real vector bundle $E \to M$ determines twisted real cohomology groups $H^\bullet(M;E)$. In differential geometry these cohomology groups are defined by extending the de Rham complex to forms with coefficients in E using the flat connection. The sorts of twistings of K-theory we consider are one-dimensional, so analogous to the case when E is a line bundle. There are also one-dimensional twistings of *integral* cohomology, determined by a *local system* $Z \to M$. This is a bundle of groups isomorphic to $\mathbb{Z}$, so is determined up to isomorphism by an element of $H^1\bigl(M;\mathrm{Aut}(\mathbb{Z})\bigr) \cong H^1(M;\mathbb{Z}/2\mathbb{Z})$, since the only nontrivial automorphism of $\mathbb{Z}$ is multiplication by -1. The twisted integral cohomology $H^\bullet(M;Z)$ may be thought of as sheaf cohomology, or defined using a cochain complex. We give a Čech description as follows. Let $\{U_i\}$ be an open covering of M and

$$g_{ij}\colon U_i \cap U_j \longrightarrow \{\pm 1\} \tag{1.1}$$

a cocycle defining the local system Z. Then an element of $H^q(M;Z)$ is represented by a collection of q-cochains $a_i \in Z^q(U_i)$ which satisfy

$$a_j = g_{ij}\, a_i \qquad \text{on } U_{ij} = U_i \cap U_j. \tag{1.2}$$

We can use any model of cochains, since the group $\mathrm{Aut}(\mathbb{Z}) \cong \{\pm 1\}$ always acts. In place of cochains we represent integral cohomology classes by maps to an Eilenberg-MacLane space $K(\mathbb{Z},q)$. The cohomology group is the set of homotopy classes of maps, but here we use honest maps as representatives. The group $\mathrm{Aut}(\mathbb{Z})$ acts on $K(\mathbb{Z},q)$. One model of $K(\mathbb{Z},0)$ is the integers, with -1 acting by multiplication. The circle is a model for $K(\mathbb{Z},1)$, and -1 acts by reflection. Using the action of $\mathrm{Aut}(\mathbb{Z})$ on $K(\mathbb{Z},q)$ and the cocycle (1.1) we build an associated bundle $\mathcal{H}^q \to M$ with fiber $K(\mathbb{Z},q)$. Equation (1.2) says that twisted cohomology classes are represented by sections of $\mathcal{H}^q \to M$; the twisted cohomology group $H^q(M;Z)$ is the set of homotopy classes of sections of $\mathcal{H}^q \to M$.

Twistings may be defined for any generalized cohomology theory; our interest is in complex K-theory. In homotopy theory one regards K as a marriage of a ring and a space (more precisely, spectrum), and it makes sense to ask for the units in K, denoted $GL_1(K)$. In the previous paragraph we used the units in integral cohomology, the group $\mathbb{Z}/2\mathbb{Z}$. For complex K-theory there is a richer group of units[2]

$$GL_1(K) \sim \mathbb{Z}/2\mathbb{Z} \times \mathbb{CP}^\infty \times BSU. \tag{1.3}$$

[2]This is a consequence of results in [DK], [S], [ASe], [AP].

In our problem the last factor doesn't enter and all the interest is in the first two, which we denote $GL_1(K)'$. As a first approximation, view K as the category of all finite dimensional $\mathbb{Z}/2\mathbb{Z}$-graded complex vector spaces. Then $\mathbb{CP}^\infty$ is the subcategory of even complex lines, and it is a group under tensor product. It acts on K by tensor product as well. The nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ in (1.3) acts on K by reversing the parity of the grading. This model is deficient since there is not an appropriate topology. One may consider instead complexes of complex vector spaces, or spaces of operators as we do below. Of course, there are good topological models of $\mathbb{CP}^\infty$, for example the space of all one-dimensional subspaces of a fixed complex Hilbert space. For a manifold M the twistings of K-theory of interest are classified up to isomorphism by

$$H^1\bigl(M; GL_1(K)'\bigr) \cong H^1(M;\mathbb{Z}/2\mathbb{Z}) \times H^3(M;\mathbb{Z}). \tag{1.4}$$

In this paper we will not encounter twistings from the first factor and will focus exclusively on the second.[3] These twistings are represented by cocycles g_{ij} with values in the space of lines, in other words by complex line bundles $L_{ij} \to U_{ij}$ which satisfy a cocycle condition. This is the data often given to define a *gerbe*.

Following [A], we present a model of twisted K-theory in terms of operators on an infinite dimensional separable complex Hilbert space H. Let $PGL(H)$ be the projective general linear group of Hilbert space. Kuiper's theorem asserts that the set of invertible transformations $GL(H)$ is contractible, whence $PGL(H) = GL(H)/\mathbb{C}^\times$ has the homotopy type of $\mathbb{CP}^\infty$. The space of Fredholm operators $\mathrm{Fred}^0(H)$ has the homotopy type of $\mathbb{Z}\times BU$, so is a classifying space for K^0, and a suitable space of self-adjoint Fredholm operators $\mathrm{Fred}^1(H)$ is a classifying space for K^1 (see [AS]). (Alternatively, we could take invertible operators of the form $1 + \text{compact}$ as a classifying space for K^1.) Recall Bott periodicity which implies that K^q depends only on the parity of q, so we need only consider $q = 0$ and $q = 1$. Then a twisting class in $H^3(M;\mathbb{Z})$ is represented by a cocycle

$$g_{ij}\colon U_{ij} \longrightarrow PGL(H),$$

and a twisted K-theory class by maps

$$a_i\colon U_i \longrightarrow \mathrm{Fred}^q(H)$$

which satisfy

$$a_j = g_{ij} a_i g_{ij}^{-1} \qquad \text{on } U_{ij}.$$

[3]In passing we remark that twistings form an abelian group, and the isomorphism (1.4) is *not* an isomorphism of groups; rather the sum of twisting classes in $H^1(M;\mathbb{Z}/2\mathbb{Z})$ has a component in $H^3(M;\mathbb{Z})$.

Associated to $\{g_{ij}\}$ is a bundle over M with fiber $\mathrm{Fred}^q(H)$, and the twisted K-groups are the sets of homotopy classes of sections.

It is important that twisted cohomology theories (such as twisted K-theory) are cohomology theories. In particular, they satisfy the Mayer-Vietoris property. Note that twisted cohomology is not a ring; rather, it is a module over untwisted cohomology. When twisted cohomology classes are multiplied the twisting class adds.

There are also twistings of equivariant cohomology theories. Suppose a compact Lie group G acts on M. Then the twistings of equivariant K-theory we consider here are classified by the equivariant cohomology group $H^3_G(M;\mathbb{Z})$.

We conclude this section by computing the twisted equivariant K-theory of SU_2, where SU_2 acts on itself by conjugation. As a preliminary, we first recall some facts about equivariant K-theory. Let $R(G)$ denote the representation ring of G. Then

$$K^q_G(pt) \cong \begin{cases} R(G), & q=0; \\ 0, & q=1. \end{cases} \tag{1.5}$$

Now suppose $H \subset G$ is a closed subgroup. Then

$$K_G(G/H) \cong K_H(pt). \tag{1.6}$$

The isomorphism is restriction to the basepoint; its inverse is the associated bundle construction. The computation below is somewhat easier in K-homology, rather than K-cohomology. Poincaré duality gives properties of equivariant K-homology analogous to (1.5) and (1.6).

The representation ring of SU_2 is a polynomial ring generated by the two-dimensional representation σ. Let $\mathbb{T} \subset SU_2$ denote the maximal torus of diagonal matrices. Then $R(\mathbb{T}) \cong \mathbb{Z}[\alpha, \alpha^{-1}]$, where α is the one-dimensional standard representation. There is a holomorphic induction map

$$\mathrm{ind}\colon R(\mathbb{T}) \longrightarrow R(SU_2), \tag{1.7}$$

also known as the Borel-Weil construction. Namely, a representation of $\mathbb{T}$ determines a holomorphic line bundle over $SU_2/\mathbb{T} \cong \mathbb{CP}^1$, and the induced representation of SU_2 is the space of its holomorphic sections. For example, $\mathrm{ind}(\alpha) = \sigma$, $\mathrm{ind}(1) = 1$, and $\mathrm{ind}(\alpha^{-1}) = 0$. Now $R(\mathbb{T})$ is an $R(SU_2)$-module—a representation of SU_2 multiplies a representation of $\mathbb{T}$ by restriction—which is free of rank 2. We take $1, \alpha^{-1}$ as generators. Note that (1.7) preserves the $R(SU_2)$-module structure.

We are ready to compute. As stated earlier we let $G = SU_2$ act on itself by conjugation. Let $\zeta \in H^3_G(G) \cong \mathbb{Z}$ be m times the generator. Cover SU_2 by the invariant sets

$$\begin{aligned} U &= SU_2 \setminus \{-1\} \\ V &= SU_2 \setminus \{+1\}. \end{aligned}$$

Then

$$U, V \sim pt$$
$$U \cap V \sim SU_2/\mathbb{T}. \tag{1.8}$$

The twisting class is represented by the m^{th} power of the hyperplane bundle on $U \cap V \sim \mathbb{CP}^1$, which is induced from the representation α^m of $\mathbb{T}$. Let $K^G_{q+\zeta}(G)$ denote the ζ-twisted equivariant K-homology in degree q. Using (1.5) and (1.8) we write the Mayer Vietoris sequence for homology as

$$0 \longrightarrow K^G_{1+\zeta}(G) \longrightarrow K^G_0(U \cap V) \longrightarrow K^G_0(U) \oplus K^G_0(V) \longrightarrow K^G_{0+\zeta}(G) \longrightarrow 0. \tag{1.9}$$

The middle terms are untwisted since the twisting class ζ restricts trivially on U, V. From (1.8), (1.5), and (1.6) we identify the middle arrow, which is a pushforward in K-homology, as the holomorphic induction

$$\begin{aligned} \mathbb{Z}[\alpha, \alpha^{-1}] &\longrightarrow \mathbb{Z}[\sigma_1] \times \mathbb{Z}[\sigma_2] \\ \rho &\longmapsto \big(\mathrm{ind}(\rho), \mathrm{ind}(\alpha^m \rho)\big). \end{aligned} \tag{1.10}$$

The image of the generators of $R(\mathbb{T})$ are expressed in terms of symmetric products of the standard representation:

$$\begin{aligned} \alpha^{-1} &\longrightarrow \big(0, \mathrm{Sym}^{m-1}(\sigma_2)\big) \\ 1 &\longmapsto \big(1, \mathrm{Sym}^m(\sigma_2)\big). \end{aligned}$$

Thus (1.10) is injective and the cokernel is easy to compute, since (1.10) is a map of $R(SU_2)$-modules:

$$\begin{aligned} K^G_{1+\zeta} &= 0, \\ K^G_{0+\zeta} &\cong R(SU_2)/\langle \mathrm{Sym}^{m-1}(\sigma_2)\rangle. \end{aligned} \tag{1.11}$$

By Poincaré duality we obtain the twisted equivariant K-cohomology, which is nonzero in odd degrees. Since (1.9) is a sequence of $R(SU_2)$-module maps, (1.11) is an isomorphism of rings: the twisted equivariant K-theory is a quotient of the representation ring. To make contact with the theorem stated in the introduction, take $m = k + 2$. Then this quotient of the representation ring is precisely the Verlinde algebra of SU_2 at level k (see [V]).

§2 Chern-Simons Theory Revisited

A characteristic property of a topological quantum field theory in n dimensions is the *gluing law*. The *partition function* is a functorial assignment

$$X^n \longmapsto Z(X) \quad \text{element of } \mathbb{C} \tag{2.1}$$

of a complex number to a closed n-manifold X. Now if the manifold X is cut by a closed hypersurface Y into a union of manifolds X_1, X_2 with boundary, then there are invariants $Z(X_1), Z(X_2)$ so that $Z(X) = Z(X_1) \cdot Z(X_2)$. But $Z(X_i)$ is not a single complex number. Rather, for each "boundary condition" α on Y we obtain complex numbers $Z(X_i)_\alpha$, and the invariant $Z(X_i)$ is the vector of these complex numbers. (In *topological* quantum field theories there is often a *finite* basis of α.) The complex vector space of these vectors is then functorially attached to the closed manifold Y:

$$Y^{n-1} \longmapsto E(Y) \quad \mathbb{C}\text{-module.} \tag{2.2}$$

The standard story ends here. But, as many people observed (see [F1], for example) it is beneficial to go further and consider hypersurfaces $S \subset Y$ which express Y as the union of submanifolds Y_1, Y_2 with boundary. Then the locality property leads us by analogy to define a vector of complex vector spaces $E(Y_i) = \bigl(E(Y_i)_\beta\bigr)$ indexed by some boundary conditions β on S. It is natural to view this as an element of a "K-module" functorially attached to S, which now has codimension 2 compared to the top dimension n:

$$S^{n-2} \longmapsto \mathcal{E}(S) \quad K\text{-module.} \tag{2.3}$$

Here we view K heuristically as the category of finite dimensional ($\mathbb{Z}/2\mathbb{Z}$-graded) complex vector spaces.

One can immediately see that a K-module is a category (whereas a $\mathbb{C}$-module is a set), but it has an additive structure as well. The additive structure on $E(Y)$ in (2.2) is crucial in quantum mechanics—it encodes the superposition of states. It stands to reason that $\mathcal{E}(S)$ should have an additive structure as well. The discussion in this section is completely heuristic, so we do not give a formal definition of K-modules; one definition in the literature, where it is called a "2-vector space," runs several pages. Rather, we give an example. Let F be a finite group. Then the category $\mathcal{E}(F)$ of complex finite dimensional representations of F is a K-module. Addition is the direct sum of representations, and scalar multiplication is tensor product with a vector space on which F acts trivially. Since we can multiply representations using the tensor product, $\mathcal{E}(F)$ is a K-algebra.

Even better, there is a sort of Hilbert space structure: the inner product of representations E_1, E_2 is the vector space of F-invariant maps $E_1 \to E_2$.

One can continue down this path, slicing and dicing manifolds of increasing codimension. This quickly leads to multicategories.[4] Such structures probably exist in any quantum field theory, since locality is a characteristic property, but the infinite dimensionality renders them of little use.

Now specialize to $n = 3$. Since every closed 1-manifold is a finite union of circles, there is only one interesting K-module $\mathcal{E} = \mathcal{E}(S^1)$ in the theory. Elementary topology in one and two dimensions gives extra structure to $\mathcal{E}$. For example, in a theory of *oriented* manifolds reflection in S^1 gives an involution on $\mathcal{E}$. The disk defines a distinguished element of $\mathcal{E}$, and the pair of pants a multiplication on $\mathcal{E}$. The standard arguments which demonstrate that the $\mathbb{C}$-module attached to S^1 in a two-dimensional field theory is a *Frobenius algebra* over $\mathbb{C}$ lead us to the conclusion that in a three-dimensional theory $\mathcal{E}$ is a "Frobenius algebra over K." We do not pretend to give a formal definition, but note that in the literature the object attached to the circle is usually called a "modular tensor category," or some close variation. In fact, there is a theorem that the entire three-dimensional topological quantum field theory can be reconstructed from this algebraic data (see [T]).

In this context the *Verlinde algebra* is the Grothendieck group of the K-algebra $\mathcal{E}$. In particular, it is an algebra over the Grothendieck group $K^\bullet(pt)$ of K.

Three-dimensional Chern-Simons theory is the most well-studied example.[5] It is defined for any compact Lie group G starting with a "level" $k \in H^4(BG;\mathbb{Z})$. One gains insight from the case when G is finite, and this was much studied. In particular, the K-module $\mathcal{E}$ has a simple description in that case. If $k = 0$ then $\mathcal{E} = \mathrm{Vect}_G(G)$, the category of G-equivariant complex vector bundles over G. The K-module structure uses pointwise direct sum and tensor product; the algebra structure is induced from the multiplication $G \times G \to G$ by pushforward. The Grothendieck group is the equivariant K-theory group $K_G(G)$. This is the simplest case of our main theorem, and it was well-known ten years ago. (The observation that equivariant K-theory enters is credited to Lusztig.) A similar picture holds if the level k is nonzero. Namely, there is a central extension of the category "G acting on G" by a transgression of k: to each pair (x, g) of elements of G, thought of as an arrow from x to gxg^{-1}, is attached a complex line, and these lines multiply suitably under composition of arrows. The K-module $\mathcal{E}$ consists of vector bundles over G suitably equivariant under this central extension. The precise statements appear in [F2], but we failed to recognize the Grothendieck group as a twisted K-theory group. That belated realization led to the main theorem.

In many cases the partition function (2.1) of a quantum field theory is formally written as a functional integral in terms of classical fields and a classical action. One of the key ideas in [F2] is to

[4]so perhaps also to multiheadaches. I define a mathematician's *category number* to be the largest n such that he/she can think about n-categories without getting a migraine.

[5]There is a subtlety we ignore: In Chern-Simons theory the category of oriented manifolds is centrally extended to oriented manifolds with "p_1-structure."

extend the notion of classical action in an n-dimensional field theory to manifolds of dimension $< n$, and to view the quantum Hilbert space (2.2) and the quantum K-module (2.3) as defined by an extended notion of the functional integral. For gauge theories with finite structure group the functional integral reduces to a finite sum, and so can be explicitly computed. This led us to the description of $\mathcal{E}$ above. For any compact group G we can best describe the extended *classical* action in terms of *differential cohomology*. This marriage of integral cohomology and differential forms first appeared in differential geometry as *Cheeger-Simons differential characters*, and also goes by the name of *smooth Deligne cohomology*. A treatment in the spirit needed here is given in [HS]. The level $\lambda \in H^4(BG;\mathbb{Z})$ lifts uniquely to a differential cohomology class in $\check{H}^4(BG)$, and we fix a particular cocycle $\check{\lambda}$ which represents it. Then if $P \to M$ is any principal G-bundle with connection A, there is a degree 4 differential cocycle $\check{\lambda}(A)$ on M which carries the information of a characteristic class of P in $H^4(M)$ and the Chern-Weil form of A in $\Omega^4(M)$. If now $M \to T$ is an oriented fiber bundle of compact manifolds with a G-connection A on M, then we can integrate $\check{\lambda}(A)$ over the fibers of $M \to T$ to construct invariants. For $X \to T$ with closed fibers of dimension 3 that integral is a cocycle for $\check{H}^1(T)$, which is simply a map $T \to \mathbb{T}$, where $\mathbb{T}$ is the circle group. This is the exponentiated Chern-Simons invariant. It is the integrand of the functional integral which defines the partition function (2.1) of quantum Chern-Simons theory [W]. Note that in defining the functional integral (over the universal family) we must extend the codomain of the Chern-Simons invariant from $\mathbb{T}$ to $\mathbb{C}$. Of course, that functional integral is a formal expression; the only mathematically rigorous definition of these invariants heretofore begin with the algebraic data in dimension one and build up from there.

Next, consider a fiber bundle $Y \to T$ whose fibers are closed oriented 2-manifolds, and suppose we have a G-connection A on Y. Now the integral of $\check{\lambda}(A)$ over the fibers is a cocycle for $\check{H}^2(T)$, which we may view as a $\mathbb{T}$-bundle with connection over T. It is the appropriate "classical action" in dimension 2 for this theory. The associated hermitian line bundle over the universal family of *flat* connections is the "prequantum line bundle" in the theory of geometric quantization. In that story one chooses a "polarization" of the symplectic manifold of flat connections, then constructs a Hilbert space of compatible 1/2-forms with coefficients in the prequantum line bundle. The most natural polarization in this case is complex, in which case we take the space of holomorphic sections of the prequantum line bundle tensored with the square root of an appropriate canonical line bundle. It is conceptually useful to regard the geometric quantization procedure as a functional integral of the classical action in dimension 2.

Finally, we pass to fiber bundles $S \to T$ whose fibers are oriented circles. Then the appropriate classical action associated to a G-connection A on S is a cocycle for $\check{H}^3(T)$, which may be thought of as a "$\mathbb{T}$-gerbe with connection". To define the quantum invariant (2.3) we let T be the universal family of connections on S^1 modulo gauge equivalence. If we fix a basepoint on S^1, then a connection is determined up to isomorphism by its holonomy; changing the basepoint conjugates the holonomy. So the category of G-connections on S^1 is equivalent to the category which expresses the action

of G on itself by conjugation. Now just as in the previous paragraph we passed from a bundle of $\mathbb{T}$-torsors to a bundle of $\mathbb{C}$-modules, here we pass from a bundle of $\mathbb{T}$-gerbes to a bundle of K-modules.[6] Formally, then, we expect $\mathcal{E}$ to be a suitable space of sections of this bundle and its Grothendieck group—the Verlinde algebra—to be the space of homotopy classes of sections. From the exposition in §1 we recognize it as a twisted equivariant K-theory group. We have almost arrived at the statement of the main theorem, but there are two caveats. First, the twisting class ζ which appears is deviates from the Chern-Simons action by a universal "adjoint shift." That shift is analogous to 1/2-form twist in geometric quantization. It is intriguing to ponder if there is a story—even heuristic—about polarizations in this context which leads to the adjoint shift. Second, the nonzero K-cohomology appears in degree $\dim G$, whereas these ideas lead one to suspect it would appear in degree 0. We have yet to find an intuitive explanation of this.

The mathematical proof of the main theorem [FHT] does not involve any of these heuristics.

[6]Let $\mathcal{T}$ denote the category of $\mathbb{T}$-torsors. A $\mathbb{T}$-gerbe is a $\mathcal{T}$-torsor, and now the analogy of linearizations is better: in the first instance we replace the group $\mathbb{T}$ by the ring $\mathbb{C}$; in the second we replace the group $\mathcal{T}$ by the ring K.

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